Extremal and Probabilistic Graph Theory
Lecture 15
April 21st, Thursday

- **Theorem 2 (Alon, Hoory and Linial).** Let G be an n-vertex graph with average degree d, where all degrees are at least 2. Then G has at least $nd(d-1)^{k-1}$ non-backtracking walks of length k, with equality if and only if G is regular and d is an integer.

  Proof. Let A be $nd \times nd$ matrix indexed by directed edges.

  $$(A)_{\vec{uv},\vec{wz}} = \begin{cases} 1, & \text{u=v and z=w} \\ 0, & \text{otherwise} \end{cases}$$

  Let P be $nd \times nd$ matrix.

  $$(P)_{\vec{uv},\vec{wz}} = \begin{cases} \frac{1}{\alpha-1}, & \text{u=v and z=w} \\ 0, & \text{otherwise} \end{cases}$$

  Then the matrix P is the transition matrix of a random walk. Let $\vec{x} = \frac{1}{nd}\vec{I}$, then $\vec{x}P = \vec{x}$. And let $N_e = |\Omega_e|$. Let

  $$N = \sum (\vec{x}_e) N_e = < \vec{x}A^{k-1}, \vec{1} > \tag{1}$$

  be the number of non-backtracking k-walks in G times $\frac{1}{nd}$.

  For walk $w = e = e_1, e_2, ..., e_k \in \Omega_e$ and $f \in e_1, e_2, ..., e_{k-1}$, denote $n_f(w)$ to be the number of times that f appears in w. Let $p(\vec{w})$ be the probability of the walk $\vec{w}$, i.e. $p(\vec{w}) = \prod_{\vec{uv}} (d_v - 1)^{-n_{\vec{uv}(w)}}(2)$. By the inequality of arithmetic-geometric means, we rewrite (1) as

  $$N = \sum_e (\vec{x})_e N_e = \sum (\vec{x}_k) \sum \frac{p(w)}{p(uv)} \geq \prod_e \prod_{w \in \Omega_e} \frac{1}{p(w)}$$

  $$= \prod_e \prod_{w \in \Omega_e} \prod_{\vec{uv}} (d_v - 1)^{-n_{\vec{uv}(w)}} = -\prod_{\vec{uv}} (d_v - 1)^{\sum_e (\vec{x})_e \sum_{w \in \Omega_e} n_{\vec{uv}(w)} p(w)}$$

  And we let the exponent of the upper result to be $E^*_{\vec{uv}}$, which is the expected number of visits to the edge $\vec{uv}$ in a backtracking random walk of length k (starting with the initial distribution $\vec{x}$). So

  $$N \geq \prod_{\vec{uv}} (d_v - 1)^{E^*_{\vec{uv}}} = \prod_{\vec{uv}} (d_v - 1)^{\sum_{i=1}^{k-1} (\vec{x}p^i)_{\vec{uv}}}$$

where $(\vec{x}p^i)_{\vec{uv}}$ equals the expected number of times that $\vec{uv}$ appears in the i^{th} edge of the random walk. Note that $\vec{x}p = \vec{x}$. So this implies that

$$N \geq \prod_{\vec{uv}} (d_v - 1)^{\sum_{i=1}^{k-1} \frac{1}{nd}} = \prod_{\vec{uv}} (d_v - 1)^{\frac{k-1}{nd}} = \left[\prod_{v} (d_v - 1)^{\frac{d_v}{nd}} \right]^{(k-1)} = \Delta^{k-1}$$

where $\Delta = \prod_v (d_v - 1)^{\frac{d_v}{nd}}$. Let $f(x) = (x-1)^x$. Then $(\log f(x))'' \geq 0$ for $x \geq 2$. So $f(x)$ is log-concave, thus

$$\log \Delta = \sum_v \frac{d_v}{nd} \log(d_v - 1) = \frac{1}{d} \cdot \left(\frac{\sum_v d_v \log(d_v - 1)}{n} \right) \geq \frac{1}{d} \cdot \frac{\sum d_v}{n} \log \left(\frac{\sum d_v}{n} - 1 \right) = \log(d-1).$$

Then $N \geq (d-1)^{k-1}$. This proves the theorem.

Moore Bound

- **Defination:** The girth of a graph G is the length of the shortest circle.
- **Defination:** Let $n_0(d, g)$ be a function such that

$$n_0(d,g) = \begin{cases} 1 + d \sum_{i=0}^{r-1} (d-1)^i, & \text{if} \quad g = 2r+1 \\ 2 \sum_{i=0}^{r-1} (d-1)^i, & \text{if} \quad g = 2r \end{cases}$$

  This function is called Moore Bound.

- **Fact:** Let G be an n-vertex d-regular graph with graph g, then $n \geq n_0(d, g)$.
- This proof is done by sketch.
- **Theorem 3.** Let G be an n-vertex graph with girth g and average degree $d \geq 2$, then $n \geq n_0(d, g)$.

  Proof. We first claim that we may assume G has no vertices of degree 0 or 1.

  Suppose there exist $v, s.t. d_v \leq 1$. Let $G' = G - v$, so G' has $n - 1$ vertices and girth more than g. And the average degree of G'

$$d' \geq \frac{nd - 2}{n - 1} \quad (\text{as} \quad d \geq 2)$$

  By induction on $|v(G')|$, we have $n - 1 \geq n_0(d', g') \geq n_0(d, g)$. This proves the Claim.

  Thus, all vertex degrees are greater than 2, and then we can apply Theorem 2.

For $g = 2r+1$, Theorem 2 shows that G has more than $nd(d-1)^{i-1}$ non-backtracking i-walks for all $i \in \{1, 2, ..., r\}$. By averaging, there exists a vertex v which is the beginning vertex of at least

$$\sum_{i=1}^{r} \frac{nd(d-1)^{i-1}}{n} = \sum_{i=1}^{r} d(d-1)^{i-1}$$

non-backtracking walks of length less than r in G. Since $g(G) \geq 2r+1$ all such walks should have distinct end-points (other than v). So $n - 1 \geq \sum_{i=1}^{r} d(d-1)^{i-1}$. Thus $n \geq 1 + \sum_{i=1}^{r} d(d-1)^{r-1} = n_0(d, 2r+1)$

For $g = 2r$, by Theorem 2, G has more than $nd(d-1)^{i-1}$ i-walks for $i \in \{1, 2, ..., r\}$. Note that $nd = 2e$. By averaging, there exists $uv \in E(G)$, which is the beginning edge ($u \to v \to ...$ and $v \to u \to ...$) of at least

$$\sum_{i=1}^{r} \frac{nd(d-1)^{i-1}}{e} = 2\sum_{i=1}^{r} (d-1)^{r-1}$$

non-backtracking walks of length less than r. Observe that all such walks should have distinct end-points. Thus $n \geq 2\sum_{i=1}^{r} (d-1)^{i-1} = n_0(d, 2r)$. This proves Theorem 3.

- **Defination:** A graph with girth g and average degree d achieving the Moore Bound is called a *Moore graph*.

  The existence of d-regular Moore graphs (for $d \geq 3$,$g \geq 5$) attracts many attentions.

- **Theorem 4 (Hoffman-Singleton).** If a d-regular Moore graph of girth exists then $d \in \{2, 3, 7, 57\}$.